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Congruences Modulo 8 for the Class Numbers of $Q(\sqrt{\pm p})$, $p \equiv 3 \pmod 4$ a Prime

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A congruence modulo 8 is proved relating the class numbers of the quadratic fields $Q(\sqrt{p})$ and $Q(\sqrt{-p})$, where p is a prime congruent to 3 modulo 4.

1. INTRODUCTION

Throughout this paper p denotes a prime (greater than 3) which is congruent to 3 modulo 4. The class number of the quadratic field $Q(\sqrt{p})$ (resp. $Q(\sqrt{-p})$) is denoted by $h(p)$ (resp. $h(-p)$). It is well known that (see, for example, [2, p. 413; 3, p. 100])

$$h(p) \equiv h(-p) \equiv 1 \pmod 2. \tag{1.1}$$

In [7] the author determined a congruence (see (4.1) below) relating $h(p)$ and $h(-p)$ modulo 4. It is the purpose of this paper to determine congruences relating these class numbers modulo 8. (The analogous problem for primes $p \equiv 1 \pmod 4$ has been treated by the author elsewhere [5, 7–11].)

2. THE FUNDAMENTAL UNIT ε_p

The fundamental unit ε_p (> 1) of the real quadratic field $Q(\sqrt{p})$ is of the form (see, for example, [4, Sect. 7])

$$\varepsilon_p = T + U\sqrt{p} = \tfrac{1}{2}(R + S\sqrt{p})^2, \tag{2.1}$$

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where T and U are positive coprime integers which satisfy

$$T \equiv 0 \pmod 2, \quad U \equiv 1 \pmod 2, \quad N(\varepsilon_p) = T^2 - pU^2 = +1, \tag{2.2}$$

and where R and S are positive coprime integers satisfying

$$\begin{aligned} R \equiv S \equiv 1 \pmod 2, \quad R^2 - pS^2 &= -2, \quad \text{if} \quad p \equiv 3 \pmod 8, \\ &= +2, \quad \text{if} \quad p \equiv 7 \pmod 8). \end{aligned} \tag{2.3}$$

Clearly T, U, R and S are related by

$$T = \tfrac{1}{2}(R^2 + pS^2), \qquad U = RS. \tag{2.4}$$

The integers R and S play a central role in everything that follows.

3. Congruences for R and S modulo 8

From (2.3) we have

$$\left(\frac{-2}{S}\right) = \left(\frac{R^2 - pS^2}{S}\right) = \left(\frac{R^2}{S}\right) = +1, \qquad \text{if} \quad p \equiv 3 \pmod 8,$$

$$\left(\frac{+2}{S}\right) = \left(\frac{R^2 - pS^2}{S}\right) = \left(\frac{R^2}{S}\right) = +1, \qquad \text{if} \quad p \equiv 7 \pmod 8,$$

so that

$$\begin{cases} S \equiv 1, 3 \pmod 8, & \text{if} \quad p \equiv 3 \pmod 8, \\ S \equiv 1, 7 \pmod 8, & \text{if} \quad p \equiv 7 \pmod 8. \end{cases} \tag{3.1}$$

Then, from (2.3) and (3.1), we obtain

LEMMA 1. (a) *If* $p \equiv 3 \pmod{16}$ *then*

$$(R, S) \equiv (1, 1), (3, 3), (5, 3) \quad \textit{or} \quad (7, 1) \pmod 8.$$

(b) *If* $p \equiv 7 \pmod{16}$ *then*

$$(R, S) \equiv (3, 1), (3, 7), (5, 1) \quad \textit{or} \quad (5, 7) \pmod 8.$$

(c) *If* $p \equiv 11 \pmod{16}$ *then*

$$(R, S) \equiv (1, 3), (3, 1), (5, 1) \quad \textit{or} \quad (7, 3) \pmod 8.$$

(d) *If* $p \equiv 15 \pmod{16}$ *then*

$$(R, S) \equiv (1, 1), (1, 7), (7, 1) \quad or \quad (7, 7) \pmod 8.$$

4. Congruences Relating $h(p)$ and $h(-p)$ (mod 4)

In [7] the author showed that

$$h(-p) \equiv h(p) + U + 1 \pmod 4. \tag{4.1}$$

Appealing to (1.1), (2.3), (2.4) and (4.1) we obtain

Lemma 2. (a) *If* $R \equiv S \pmod 4$

$$h(-p) + h(p) \equiv 0 \pmod 4.$$

(b) *If* $R \equiv -S \pmod 4$

$$h(-p) - h(p) \equiv 0 \pmod 4.$$

5. Congruences Relating $h(p)$ and $h(-p)$ (mod 8)—Statement of Main Theorem

It is the purpose of this paper to prove, by extending the ideas used in [7], a more precise form of Lemma 2. We prove

Theorem. (a) *If* $R \equiv S \pmod 4$

$$h(-p) + h(p) \equiv R + S + 2(-1)^{(p-3)/4} \pmod 8.$$

(b) *If* $R \equiv -S \pmod 4$

$$h(-p) - h(p) \equiv R - S - 2 \pmod 8.$$

The proof of this theorem is completed in Section 12, after a number of lemmas are proved in Sections 6–11. It uses the ideas of [7] but is much more complicated in its details.

6. The polynomials $F_+(z)$ and $F_-(z)$

We set $\rho = \exp(2\pi i/p)$ and, for z a complex variable, we define (as in [7])

$$F_+(z) = \prod_{\substack{j=1 \\ \left(\frac{j}{p}\right) = +1}}^{p-1} (z - \rho^j), \qquad F_-(z) = \prod_{\substack{j=1 \\ \left(\frac{j}{p}\right) = -1}}^{p-1} (z - \rho^j), \tag{6.1}$$

so that

$$F(z) = F_+(z)\,F_-(z) = \prod_{j=1}^{p-1} (z - \rho^j) = \frac{z^p - 1}{z - 1} = z^{p-1} + z^{p-2} + \cdots + 1. \tag{6.2}$$

It is easily checked that

$$F(1) = p, \qquad F(-1) = 1, \qquad F(\pm i) = \pm i, \tag{6.3}$$

and

$$\begin{gathered} F'(1) = \tfrac{1}{2}p(p-1), \qquad F'(-1) = -\tfrac{1}{2}(p-1), \\ F'(\pm i) = \tfrac{1}{2}(p-1) \pm \tfrac{1}{2}(p+1)\,i. \end{gathered} \tag{6.4}$$

7. Evaluation of $F_\pm(-1)$ and $F_\pm(\pm i)$

Throughout the rest of the paper the convention $\sqrt{-p} = i\sqrt{p}$ is used. We prove

Lemma. 3.

$$F_+(1) = (-1^{1/2(h(-p)+1)}\sqrt{-p},$$

$$F_-(1) = (-1)^{1/2(h(-p)-1)}\sqrt{-p},$$

$$F_+(-1) = F_-(-1) = (-1)^{1/4(p-3)},$$

$$F_+(i) = \left\{ \begin{matrix} \omega^3(-1)^{1/2(h(-p)+1)}\,\varepsilon_p^{-h(p)/2}, \\ \text{if} \quad p \equiv 3 \pmod 8 \\ \omega^5\varepsilon_p^{-h(p)/2} \qquad \text{if} \quad p \equiv 7 \pmod 8 \end{matrix} \right\},$$

$$F_-(i) = \left\{ \begin{matrix} \omega^7(-1)^{1/2(h(-p)+1)}\,\varepsilon_p^{h(p)/2}, \\ \text{if} \quad p \equiv 3 \pmod 8 \\ \omega^5\varepsilon_p^{h(p)/2}, \qquad \text{if} \quad p \equiv 7 \pmod 8 \end{matrix} \right\},$$

$$F_+(-i)=\begin{Bmatrix} \omega(-1)^{1/2(h(-p)+1)}\,\varepsilon_p^{h(p)/2}, \\ \quad\text{if}\quad p\equiv 3 \pmod 8 \\ \omega^3\varepsilon_p^{h(p)/2}, \qquad \text{if}\quad p\equiv 7 \pmod 8 \end{Bmatrix},$$

$$F_-(-i)=\begin{Bmatrix} \omega^5(-1)^{1/2(h(-p)+1)}\,\varepsilon_p^{-h(p)/2}, \\ \quad\text{if}\quad p\equiv 3 \pmod 8 \\ \omega^3\varepsilon_p^{-h(p)/2}, \qquad \text{if}\quad p\equiv 7 \pmod 8 \end{Bmatrix},$$

where $\omega=(1+i)/\sqrt{2}$ *is an eighth root of unity.*

Proof. We just give the details of the evaluation of $F_-(i)$ as the other cases are similar. From (6.1) we have (where the dash indicates that j is restricted to satisfy $(\frac{j}{p})=-1$)

$$F_-(i)=\prod_{j=1}^{p-1}{}'\,(i-\rho^j)=i^{1/2(p-1)}\prod_{j=1}^{p-1}{}'\,(1+i\rho^j).$$

As $i\rho^j$ $(1\leqslant j\leqslant p-1)$ is a root of unity (not equal to 1), we have

$$\gamma_j=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,i^n\rho^{jn}}{n}=\log(1+i\rho^j) \qquad (j=1,2,\ldots,p-1)$$

and so

$$\exp(\gamma_j)=1+i\rho^j.$$

Thus we have

$$\prod_{j=1}^{p-1}{}'\,(1+i\rho^j)=\prod_{j=1}^{p-1}{}'\,\exp(\gamma_j)=\exp\left(\sum_{j=1}^{p-1}{}'\,\gamma_j\right).$$

Now

$$\begin{aligned}\sum_{j=1}^{p-1}{}'\,\gamma_j&=\sum_{j=1}^{p-1}{}'\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,i^n\rho^{jn}}{n}\\ &=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,i^n}{n}\sum_{j=1}^{p-1}{}'\,\rho^{jn}\\ &=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,i^n}{n}\left\{p-1-\left(\frac{n}{p}\right)\sqrt{-p}-\left(\frac{n}{p}\right)^2 p\right\},\end{aligned}$$

where we have again used the evaluation of the Gauss sum in the form which includes $n\equiv 0 \pmod p$. After a little simplification we obtain

$$\sum_{j=1}^{p-1}{}'\,\gamma_j=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-i)^n-i^n}{n}+\frac{1}{2}\sqrt{-p}\sum_{n=1}^{\infty}\frac{(-i)^n}{n}\left(\frac{n}{p}\right).$$

Now

$$\sum_{n=1}^{\infty} \frac{(-i)^n - i^n}{n} = -2i \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = -\frac{\pi i}{2}$$

and

$$\sum_{n=1}^{\infty} \frac{(-i)^n}{n} \left(\frac{n}{p}\right) = \frac{1}{2} \left(\frac{2}{p}\right) \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{n}{p}\right) - i \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{2n+1}{p}\right).$$

From Dirichlet's class number formulae for $Q(\sqrt{-p})$ and $Q(\sqrt{p})$ (see, for example, [1, p. 343]), we deduce easily that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{n}{p}\right) = \frac{\pi}{\sqrt{p}} \left(\left(\frac{2}{p}\right) - 1\right) h(-p)$$

and

$$\sum_{n=0}^{\infty} \left(\frac{2n+1}{p}\right) \frac{(-1)^n}{2n+1} = \frac{h(p)}{\sqrt{p}} \log \varepsilon_p,$$

so that

$$\sum_{n=1}^{\infty} \frac{(-i)^n}{n} \left(\frac{n}{p}\right) = \frac{\pi h(-p)}{2\sqrt{p}} \left(1 - \left(\frac{2}{p}\right)\right) - \frac{ih(p)}{\sqrt{p}} \log \varepsilon_p.$$

Hence

$$\sum_{j=1}^{p-1}{}' \gamma_j = -\frac{\pi i}{4} + \frac{\pi i h(-p)}{4} \left(1 - \left(\frac{2}{p}\right)\right) + \frac{h(p)}{2} \log \varepsilon_p$$

and so

$$\prod_{j=1}^{p-1}{}' (1 + i\rho^j) = \omega^{-1} i^{1/2(1-(2/p))h(-p)} \varepsilon_p^{h(p)/2}$$

giving

$$\begin{aligned} F_-(i) &= \omega^{-1} i^{(p-1)/2 + 1/2(1-(2/p))h(-p)} \varepsilon_p^{h(p)/2} \\ &= \omega^7 (-1)^{1/2(h(-p)+1)} \varepsilon_p^{h(p)/2}, \qquad \text{if } p \equiv 3 \pmod 8, \\ &= \omega^5 \varepsilon_p^{h(p)/2}, \qquad \text{if } p \equiv 7 \pmod 8). \end{aligned}$$

The value of $F_+(i)$ now follows from (6.2) and (6.3). For the values of $F_\pm(-i)$ we have only to note that

$$F_\pm(-i) = \prod_{\substack{j=1 \\ \left(\frac{j}{p}\right) = \pm 1}}^{p-1} (-i - \rho^j) = \prod_{\substack{j=1 \\ \left(\frac{j}{p}\right) = \mp 1}}^{p-1} (-i - \rho^{-j}) = \overline{F_\mp(i)}.$$

8. The Polynomials $Y(z)$ and $Z(z)$

$F_\pm(z)$ are polynomials in z of degree $\frac{1}{2}(p-1)$ with coefficients in the ring of integers of $Q(\sqrt{-p})$ (see [3]). Hence we can write

$$F_+(z) = \tfrac{1}{2}(Y(z) - Z(z)\sqrt{-p}), \qquad F_-(z) = \tfrac{1}{2}(Y(z) + Z(z)\sqrt{-p}), \quad (8.1)$$

where $Y(z)$ and $Z(z)$ are polynomials with rational integer coefficients. Clearly we have

$$Y(z) = F_-(z) + F_+(z), \qquad Z(z) = \frac{F_-(z) - F_+(z)}{\sqrt{-p}}. \quad (8.2)$$

Taking $z = 1, -1, i$ in (8.2) and appealing to Lemma 3 we obtain

$$Y(1) = 0, \qquad Z(1) = 2(-1)^{1/2(h(-p)-1)} \quad (8.3)$$

$$Y(-1) = 2(-1)^{1/4(p-3)}, \qquad Z(-1) = 0, \quad (8.4)$$

$$\left.\begin{aligned} Y(i) &= \omega^3(-1)^{1/2(h(-p)-1)}(\varepsilon_p^{h(p)/2} - \varepsilon_p^{-h(p)/2}), && \text{if } p \equiv 3 \pmod 8, \\ &= \omega^5(\varepsilon_p^{h(p)/2} + \varepsilon_p^{-h(p)/2}), && \text{if } p \equiv 7 \pmod 8, \\ Z(i) &= \omega^3(-1)^{1/2(h(-p)-1)}(\varepsilon_p^{h(p)/2} + \varepsilon_p^{-h(p)/2})/\sqrt{-p}, && \text{if } p \equiv 3 \pmod 8, \\ &= \omega^5(\varepsilon_p^{h(p)/2} - \varepsilon_p^{-h(p)/2})/\sqrt{-p}, && \text{if } p \equiv 7 \pmod 8. \end{aligned}\right\} \quad (8.5)$$

Since (using (2.1) and 2.3))

$$\varepsilon_p^{h(p)/2} = (T + U\sqrt{p})^{(h(p)-1)/2}\,\frac{(R + S\sqrt{p})}{\sqrt{2}}$$

and

$$\varepsilon_p^{-h(p)/2} = (T - U\sqrt{p})^{(h(p)-1)/2}\,\frac{(R - S\sqrt{p})}{\sqrt{2}}\,(-1)^{(p+1)/4},$$

we see from (8.5) that

$$\begin{aligned} Y(i) &= A_3(1-i), && \text{if} \quad p \equiv 3 \pmod 8, \\ &= A_7(1+i), && \text{if} \quad p \equiv 7 \pmod 8, \\ Z(i) &= -B_3(1+i), && \text{if} \quad p \equiv 3 \pmod 8, \\ &= B_7(1-i), && \text{if} \quad p \equiv 7 \pmod 8, \end{aligned} \tag{8.6}$$

for rational integers A_3, B_3, A_7, B_7 (see [3, Eq. (10)]). From (6.2) and (8.1) we have [3, Eq. (6)])

$$Y(z)^2 + pZ(z)^2 = 4F(z). \tag{8.7}$$

Taking $z = i$ in (8.7), and using (6.3) and (8.6), we obtain (see [3, Eq. (12)])

$$\begin{cases} A_3^2 - pB_3^2 = -2, & \text{if} \quad p \equiv 3 \pmod 8, \\ A_7^2 - pB_7^2 = +2, & \text{if} \quad p \equiv 7 \pmod 8). \end{cases} \tag{8.8}$$

Clearly (8.8) shows that A_3, B_3, A_7, B_7 are all odd.

9. The Polynomials $Y'(z)$ and $Z'(z)$

Differentiating (8.7) with respect to z, we obtain

$$Y(z)\, Y'(z) + pZ(z)\, Z'(z) = 2F'(z) \tag{9.1}$$

(see [3, Eq. (9)]). In [7, Eq. (14)] the following identity of Liouville was noted

$$Z(z)\, Y'(z) - Y(z)\, Z'(z) = 2G(z), \tag{9.2}$$

where

$$G(z) = \frac{1}{z-1} \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) z^{p-1-j}. \tag{9.3}$$

Solving (9.1) and (9.2) simultaneously for $Y'(z)$ and $Z'(z)$, we obtain (making use of (8.7))

$$\begin{cases} Y'(z) = \dfrac{F'(z)\, Y(z) + pG(z)\, Z(z)}{2F(z)}, \\ Z'(z) = \dfrac{-G(z)\, Y(z) + F'(z)\, Z(z)}{2F(z)}. \end{cases} \tag{9.4}$$

Since

$$G(1) = ph(-p), \qquad \left(\text{recalling } \sum_{j=1}^{p-1} j\left(\frac{j}{p}\right) = -ph(-p)\right), \tag{9.5}$$

$$G(-1) = \left\{1 - 2\left(\frac{2}{p}\right)\right\} h(-p), \quad \left(\text{using } \sum_{j=1}^{1/2(p-1)} \left(\frac{j}{p}\right) = \left(2 - \left(\frac{2}{p}\right)\right) h(-p)\right), \tag{9.6}$$

$$G(i) = \left\{2 - \left(\frac{2}{p}\right)\right\} h(-p), \qquad (\text{see [7, Eq. (17)]}) \tag{9.7}$$

we have

$$Y'(1) = (-1)^{1/2(h(-p)-1)}\, ph(-p), \qquad Z'(1) = (-1)^{1/2(h(-p)-1)}\, \frac{p-1}{2}, \tag{9.8}$$

$$Y'(-1) = \left(\frac{2}{p}\right)\frac{p-1}{2}, \qquad Z'(-1) = \left\{\left(\frac{2}{p}\right) - 2\right\} h(-p), \tag{9.9}$$

$$\begin{aligned}
Y'(i) &= \frac{1}{2}(A_3 - 3ph(-p)\,B_3) + \frac{i}{2}(-pA_3 + 3ph(-p)\,B_3), && \text{if } p \equiv 3 \pmod 8,\\
&= \frac{1}{2}(pA_7 - ph(-p)\,B_7 + \frac{i}{2}(A_7 - ph(-p)\,B_7), && \text{if } p \equiv 7 \pmod 8,\\
Z'(i) &= \frac{1}{2}(3h(-p)\,A_3 - pB_3) + \frac{i}{2}(3h(-p)\,A_3 - B_3), && \text{if } p \equiv 3 \pmod 8,\\
&= \frac{1}{2}(-h(-p)\,A_7 + B_7) + \frac{i}{2}(h(-p)\,A_7 - pB_7), && \text{if } p \equiv 7 \pmod 8.
\end{aligned} \tag{9.10}$$

10. $h(-p)$ Determined Modulo 8

In [7, Eq. (20)] we showed that

$$\begin{aligned}
h(-p) &\equiv -A_3B_3 \pmod 4, && \text{if } \ p \equiv 3 \pmod 8,\\
&\equiv -A_7B_7 \pmod 4, && \text{if } \ p \equiv 7 \pmod 8.
\end{aligned} \tag{10.1}$$

Our next task in this paper is to extend (10.1) to a congruence modulo 8. We prove

LEMMA 4.

$$\begin{aligned} h(-p) &\equiv A_3 B_3 + 2B_3 \pmod 8, && \text{if} \quad p \equiv 3 \pmod 8, \\ &\equiv A_7 B_7 + 2B_7 \pmod 8, && \text{if} \quad p \equiv 7 \pmod 8. \end{aligned}$$

Proof. It is known that $Y(z)$ and $Z(z)$ have the form (see [7, Eq. (7)])

$$Y(z) = \sum_{n=0}^{(p-3)/4} a_n (z^{(p-1)/2-n} - z^n), \quad Z(z) = \sum_{n=0}^{(p-3)/4} b_n (z^{(p-1)/2-n} + z^n), \tag{10.2}$$

where the a_n and b_n are integers. (This is a consequence of the easily proved result $z^{(p-1)/2} F_{\pm}(\frac{1}{z}) = -F_{\mp}(z)$ $(z \neq 0)$.) Differentiating (10.2) with respect to z we obtain (see [7, Eq. (8)])

$$\begin{cases} Y'(z) = \displaystyle\sum_{n=0}^{(p-3)/4} a_n \left(\left(\frac{p-1}{2} - n \right) z^{(p-3)/2-n} - n z^{n-1} \right), \\ Z'(z) = \displaystyle\sum_{n=0}^{(p-3)/2} b_n \left(\left(\frac{p-1}{2} - n \right) z^{(p-3)/2-n} + n z^{n-1} \right). \end{cases} \tag{10.3}$$

We now consider two cases according as $p \equiv 3$ or $7 \pmod 8$, just providing the details when $p \equiv 3 \pmod 8$. With $p = 8l + 3$, taking $z = i$ in (10.3) we obtain

$$\begin{aligned} Y'(i) = {} & \\ & \Bigg\{ \sum_{0 \leqslant m \leqslant l/2} a_{4m}(4l - 4m + 1) - \sum_{0 \leqslant m \leqslant (l-1)/2} a_{4m+1}(4m + 1) \\ & + \sum_{0 \leqslant m \leqslant (l-1)/2} a_{4m+2}(4m - 4l + 1) + \sum_{0 \leqslant m < l/2 - 1} a_{4m+3}(4m + 3) \Bigg\} \\ & + i \Bigg\{ \sum_{0 \leqslant m \leqslant l/2} a_{4m} 4m - \sum_{0 \leqslant m \leqslant (l-1)/2} a_{4m+1} 4(l - m) - \sum_{0 \leqslant m \leqslant (l-1)/2} a_{4m+2}(4m + 2) \\ & + \sum_{0 \leqslant m \leqslant l/2 - 1} a_{4m+3}(4l - 4m - 2) \Bigg\}. \end{aligned}$$

Hence from (9.10) we have

$$\begin{aligned} \tfrac{1}{2}(A_3 - 3ph(-p)\, B_3) = {} & \sum_{0 \leqslant m \leqslant l/2} a_{4m} - \sum_{0 \leqslant m \leqslant (l-1)/2} a_{4m+1} + \sum_{0 \leqslant m \leqslant (l-1)/2} a_{4m+2} \\ & - \sum_{0 \leqslant m \leqslant l/2 - 1} a_{4m+3} \pmod 4 \end{aligned}$$

$$\begin{aligned}&\equiv \sum_{0\leqslant m\leqslant l} a_{2m} - \sum_{0\leqslant m\leqslant l-1} a_{2m+1} \pmod 4\\ &= -\tfrac{1}{2}Y(-1) \qquad (\text{by } (10.2))\\ &= -1 \qquad (\text{by } (8.4))\end{aligned}$$

so

$$A_3 - 3ph(-p)\,B_3 \equiv -2 \pmod 8,$$

and thus

$$h(-p) \equiv A_3B_3 + 2B_3 \pmod 8.$$

Similarly, with $p = 8l + 7$, we obtain

$$h(-p) \equiv A_7B_7 + 2B_7 \pmod 8.$$

11. Consideration of $(R + S\sqrt{p})^{h(p)}$

From (8.1) and (8.6) we have

$$\begin{aligned}F_-(i) &= \frac{1}{2}\,(Y(i) + Z(i)\sqrt{-p})\\
&= \begin{cases}\dfrac{1}{2}\,(A_3(1-i) - B_3(1+i)\,i\sqrt{p}), & \text{if } p\equiv 3 \pmod 8,\\[2mm] \dfrac{1}{2}\,(A_7(1+i) + B_7(1-i)\,i\sqrt{p}), & \text{if } p\equiv 7 \pmod 8,\end{cases}\\
&= \begin{cases}\dfrac{1-i}{2}\,(A_3 + B_3\sqrt{p}), & \text{if } p\equiv 3 \pmod 8,\\[2mm] \dfrac{1+i}{2}\,(A_7 + B_7\sqrt{p}), & \text{if } p\equiv 7 \pmod 8,\end{cases}\\
&= \begin{cases}\dfrac{\omega^7}{\sqrt{2}}\,(A_3 + B_3\sqrt{p}), & \text{if } p\equiv 3 \pmod 8,\\[2mm] \dfrac{\omega}{\sqrt{2}}\,(A_7 + B_7\sqrt{p}), & \text{if } p\equiv 7 \pmod 8.\end{cases}\end{aligned}$$

On the other hand, from Lemma 3, we have

$$F_-(i) = \begin{cases}\omega^7(-1)^{1/2(h(-p)+1}\,\varepsilon_p^{h(p)/2}, & \text{if } p\equiv 3 \pmod 8,\\ \omega^5\varepsilon_p^{h(p)/2}, & \text{if } p\equiv 7 \pmod 8,\end{cases}$$

$$= \begin{cases} \dfrac{\omega^7}{2^{h(p)/2}} (-1)^{1/2(h(-p)+1)} (R + S\sqrt{p})^{h(p)}, & \text{if} \quad p \equiv 3 \pmod 8, \\ \dfrac{\omega^5}{2^{h(p)/2}} (R + S\sqrt{p})^{h(p)}, & \text{if} \quad p \equiv 7 \pmod 8. \end{cases}$$

Equating these two expressions for $F_-(i)$ we obtain

LEMMA 5.

$$\begin{aligned} &(R + S\sqrt{p})^{h(p)} \\ &\quad = (-1)^{(h(-p)+1)/2}\, 2^{(h(p)-1)/2} (A_3 + B_3\sqrt{p}), \qquad \text{if} \quad p \equiv 3 \pmod 8, \\ &\quad = -2^{(h(p)-1)/2} (A_7 + B_7\sqrt{p}), \qquad \text{if} \quad p \equiv 7 \pmod 8. \end{aligned}$$

We next expand $(R + S\sqrt{p})^{h(p)}$ in such a way that, using Lemma 5, we can obtain A_3, B_3, A_7, B_7 as polynomials in R and S with integral coefficients. This is done by using the following well-known identity (see, for example, [6])

$$\alpha^{2m+1} + \beta^{2m+1} = \sum_{j=0}^{m} (-1)^j \frac{2m+1}{2m+1-j} \binom{2m+1-j}{j} (\alpha+\beta)^{2m+1-2j} (\alpha\beta)^j. \tag{11.1}$$

Taking $\alpha = R + S\sqrt{p}$ and $\beta = \pm(R - S\sqrt{p})$ in (11.1) and adding, we obtain (as $R^2 - pS^2 = (-1)^{(p+1)/4}\, 2$)

$$\begin{aligned} &(R + S\sqrt{p})^{2m+1} \\ &\quad = \sum_{j=0}^{m} (-1)^{((p-3)/4)j} \frac{2m+1}{2m+1-j} \binom{2m+1-j}{j} 2^{2m-j} R^{2(m-j)+1} \\ &\qquad + \sqrt{p} \sum_{j=0}^{m} (-1)^{((p+1)/4)j} \frac{2m+1}{2m+1-j} \binom{2m+1-j}{j} \\ &\qquad \times 2^{2m-j} p^{m-j} S^{2(m-j)+1}. \end{aligned}$$

Changing the summation variable from j to $k = m - j$, and noting that

$$\frac{2m+1}{2m+1-j} \binom{2m+1-j}{j} = \frac{2m+1}{m+k+1} \binom{m+k+1}{m-k} = \frac{2m+1}{2k+1} \binom{m+k}{m-k},$$

we obtain

$$(R+S\sqrt{p})^{2m+1} = \sum_{k=0}^{m} (-1)^{((p-3)/4)(m-k)} \frac{2m+1}{2k+1} \binom{m+k}{m-k} 2^{m+k} R^{2k+1}$$
$$+\sqrt{p} \sum_{k=0}^{m} (-1)^{((p+1)/4)(m-k)} \frac{2m+1}{2k+1} \binom{m+k}{m-k}$$
$$\times 2^{m+k} p^k S^{2k+1}.$$

Taking $m = \frac{1}{2}(h(p)-1)$ in this identity and applying Lemma 5 we obtain

LEMMA 6. (i) $p \equiv 3 \pmod 8$

$$A_3 = (-1)^{(h(-p)+1)/2} h(p) \sum_{k=0}^{(h(p)-1)/2} \frac{2^k}{2k+1} \binom{(h(p)+2k-1)/2}{(h(p)-2k-1)/2} R^{2k+1},$$

$$B_3 = (-1)^{(h(-p)+h(p))/2} h(p) \sum_{k=0}^{(h(p)-1)/2} \frac{(-1)^k 2^k}{2k+1}$$
$$\times \binom{(h(p)+2k-1)/2}{(h(p)-2k-1)/2} p^k S^{2k+1}.$$

(ii) *If* $p \equiv 7 \pmod 8$

$$A_7 = (-1)^{(h(p)+1)/2} h(p) \sum_{k=0}^{(h(p)-1)/2} \frac{(-1)^k 2^k}{2k+1} \binom{(h(p)+2k-1)/2}{(h(p)-2k-1)/2} R^{2k+1},$$

$$B_7 = -h(p) \sum_{k=0}^{(h(p)-1)/2} \frac{2^k}{2k+1} \binom{(h(p)+2k-1)/2}{(h(p)-2k-1)/2} p^k S^{2k+1}.$$

Reducing the expressions in Lemma 6 modulo 8, we obtain (using 4.2)).

LEMMA 7. (i) *If* $p \equiv 3 \pmod 8$ *then*

$$\begin{aligned}
(A_3, B_3) &\equiv (7(-1)^{(R+S)/2} R,\ 7(-1)^{(R+S)/2} S) \pmod 8, \\
&\qquad \text{if} \quad h(p) \equiv 1 \pmod 8, \\
&\equiv (5(-1)^{(R+S)/2} R,\ 3(-1)^{(R+S)/2} S) \pmod 8, \\
&\qquad \text{if} \quad h(p) \equiv 3 \pmod 8, \\
&\equiv (5(-1)^{(R+S)/2} R,\ 5(-1)^{(R+S)/2} S) \pmod 8, \\
&\qquad \text{if} \quad h(p) \equiv 5 \pmod 8, \\
&\equiv (7(-1)^{(R+S)/2} R,\ (-1)^{(R+S)/2} S) \pmod 8, \\
&\qquad \text{if} \quad h(p) \equiv 7 \pmod 8.
\end{aligned}$$

(ii) *If* $p \equiv 7 \pmod 8$ *then*

$$\begin{aligned}(A_7, B_7) &\equiv (7R, 7S) \pmod 8, && \text{if} \quad h(p) \equiv 1 \pmod 8,\\ &\equiv (R, 7S) \pmod 8, && \text{if} \quad h(p) \equiv 3 \pmod 8,\\ &\equiv (R, S) \pmod 8, && \text{if} \quad h(p) \equiv 5 \pmod 8,\\ &\equiv (7R, S) \pmod 8, && \text{if} \quad h(p) \equiv 7 \pmod 8.\end{aligned}$$

The next lemma tells us the congruence classes of (A_3, B_3) and (A_7, B_7) modulo 8.

LEMMA 8. (a) *If* $p \equiv 3 \pmod{16}$ *then*

$$(A_3, B_3) = (1, 1),\ (1, 7),\ (3, 3) \text{ or } (3, 5) \pmod 8.$$

(b) *If* $p \equiv 7 \pmod{16}$ *then*

$$(A_7, B_7) \equiv (3, 1),\ (3, 7),\ (5, 1) \text{ or } (5, 7) \pmod 8.$$

(c) *If* $p \equiv 11 \pmod{16}$ *then*

$$(A_3, B_3) \equiv (5, 1),\ (5, 7),\ (7, 3) \text{ or } (7, 5) \pmod 8.$$

(d) *If* $p \equiv 15 \pmod{16}$ *then*

$$(A_7, B_7) \equiv (1, 1),\ (1, 7),\ (7, 1) \text{ or } (7, 7) \pmod 8.$$

Proof. We just provide the details for $p \equiv 3 \pmod{16}$. By Lemma 1 we have

$$(-1)^{(R+S)/2} R \equiv 5 \text{ or } 7 \pmod 8,$$

and by Lemma 7 we have

$$A_3 \equiv 5(-1)^{(R+S)/2} R \text{ or } 7(-1)^{(R+S)/2} R \pmod 8,$$

so

$$A_3 \equiv 1 \text{ or } 3 \pmod 8.$$

If $A_3 \equiv 1 \pmod 8$, $\quad B_3^2 \equiv 11pB_3^2 \equiv 11(A_3^2 + 2) \equiv 1 \pmod{16}$,

$$B_3 \equiv 1, 7 \pmod 8.$$

If $A_3 \equiv 3 \pmod 8$, $\quad B_3^2 \equiv 11pB_3^2 \equiv 11(A_3^2 + 2) \equiv 9 \pmod{16}$,

$$B_3 \equiv 3, 5 \pmod 8.$$

Putting together Lemmas 4 and 8 we obtain

LEMMA 9. (a) *If* $p \equiv 3 \pmod{16}$ *then*

$$\begin{aligned} h(-p) &\equiv 1 \pmod 8, && \textit{if} \quad (A_3, B_3) \equiv (3, 5) \pmod 8,\\ &\equiv 3 \pmod 8, && \textit{if} \quad (A_3, B_3) \equiv (1, 1) \pmod 8,\\ &\equiv 5 \pmod 8, && \textit{if} \quad (A_3, B_3) \equiv (1, 7) \pmod 8,\\ &\equiv 7 \pmod 8, && \textit{if} \quad (A_3, B_3) \equiv (3, 3) \pmod 8. \end{aligned}$$

(b) *If* $p \equiv 7 \pmod{16}$ *then*

$$\begin{aligned} h(-p) &\equiv 1 \pmod 8, && \textit{if} \quad (A_7, B_7) \equiv (5, 7) \pmod 8,\\ &\equiv 3 \pmod 8, && \textit{if} \quad (A_7, B_7) \equiv (3, 7) \pmod 8,\\ &\equiv 5 \pmod 8, && \textit{if} \quad (A_7, B_7) \equiv (3, 1) \pmod 8,\\ &\equiv 7 \pmod 8, && \textit{if} \quad (A_7, B_7) \equiv (5, 1) \pmod 8. \end{aligned}$$

(c) *If* $p \equiv 11 \pmod{16}$ *then*

$$\begin{aligned} h(-p) &\equiv 1 \pmod 8, && \textit{if} \quad (A_3, B_3) \equiv (5, 7) \pmod 8,\\ &\equiv 3 \pmod 8, && \textit{if} \quad (A_3, B_3) \equiv (7, 3) \pmod 8)),\\ &\equiv 5 \pmod 8, && \textit{if} \quad (A_3, B_3) \equiv (7, 5) \pmod 8,\\ &\equiv 7 \pmod 8, && \textit{if} \quad (A_3, B_3) \equiv (5, 1) \pmod 8. \end{aligned}$$

(d) *If* $p \equiv 15 \pmod{16}$ *then*

$$\begin{aligned} h(-p) &\equiv 1 \pmod 8, && \textit{if} \quad (A_7, B_7) \equiv (7, 1) \pmod 8,\\ &\equiv 3 \pmod 8, && \textit{if} \quad (A_7, B_7) \equiv (1, 1) \pmod 8,\\ &\equiv 5 \pmod 8, && \textit{if} \quad (A_7, B_7) \equiv (1, 7) \pmod 8,\\ &\equiv 7 \pmod 8, && \textit{if} \quad (A_7, B_7) \equiv (7, 7) \pmod 8. \end{aligned}$$

12. PROOF OF THEOREM

The theorem now follows easily from Lemmas 1, 7 and 9. We just give the details when $p \equiv 3 \pmod{16}$, as the other cases can be treated similarly (see Table).

We remark that tables of $h(p)$, $h(-p)$ and ε_p show that every one of the 64 possible cases of $(h(p), R, S) \pmod 8$ actually occurs.

Next we give a single numerical example to illustrate the theorem. We take $p = 9539 \equiv 3 \pmod{16}$. In this case

$$\varepsilon_p = \tfrac{1}{2}(293 + 3\sqrt{9539})^2,$$

TABLE I

$h(p)$ (mod 8)	R(mod 8) (from Lemma 1)	S(mod 8)	A_3(mod 8) (from Lemma 7)	B_3(mod 8)	$h(-p)$(mod 8) (from Lemma 9)	$h(-p) + (-1)^{(R-S)/2} h(p)$ (mod 8)
1	1	1	1	1	3	4
1	3	3	3	3	7	0
1	5	3	3	5	1	0
1	7	1	1	7	5	4
3	1	1	3	5	1	4
3	3	3	1	7	5	0
3	5	3	1	1	3	0
3	7	1	3	3	7	4
5	1	1	3	3	7	4
5	3	3	1	1	3	0
5	5	3	1	7	5	0
5	7	1	3	5	1	4
7	1	1	1	7	5	4
7	3	3	3	5	1	0
7	5	3	3	3	7	0
7	7	1	1	1	3	4

so $R = 293 \equiv 5 \pmod 8$, $S \equiv 3 \pmod 8$. Thus by the theorem $h(-p) - h(p) \equiv 0 \pmod 8$. Indeed $h(-p) = 55$, $h(p) = 7$.

Finally we remark that as (appealing to (2.3) and (2.4))

$$\begin{aligned} \left(\frac{T}{U}\right) &= \left(\frac{-1}{S}\right), \qquad \text{if} \quad p \equiv 3 \pmod 8, \\ &= \left(\frac{-1}{R}\right), \qquad \text{if} \quad p \equiv 7 \pmod 8, \end{aligned}$$

the theorem can also be formulated in the form

THEOREM′.

$$h(-p) \equiv h(p) \left(2 + pU - 2\left(\frac{T}{U}\right)\right) \pmod 8.$$

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